Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 3, pp. 407–416, 2008

EFFECT OF BULK VISCOSITY ON KELVIN–HELMHOLTZ INSTABILITY

Yu. N. Grigor'ev[1] **and I. V. Ershov**[2]

UDC 532.5:532.517.4

An energy functional leading to a resolvable variational problem for determining the critical Reynolds number of laminar–turbulent transition $\mathrm{Re}_{\mathrm{cr}}$ *is constructed within the framework of the nonlinear energy stability theory of compressible flows. Asymptotic estimates containing the characteristic dependence* $\mathrm{Re}_{\mathrm{cr}} \sim \sqrt{\alpha + 4/3}$ $(\alpha = \eta_b/\eta)$ *in the main order are obtained for the stability of various modes of Couette compressible gas flow. The asymptotics considered are long-wave approximations. This suggests that the obtained dependence describes the effect of bulk viscosity on the large-scale vortex structures characteristic of Kelvin–Helmholtz instability.*

Key words: *hydrodynamic stability, energy theory, compressible gas flow, bulk viscosity, laminar–turbulent transition, critical Reynolds number.*

Introduction. The dissipation effect in molecular gases, which is manifested in anomalous absorption of high-frequency sound, has been known since the 1930s [1]. Recently, this effect has been studied in aerodynamics in order to use it to retard laminar–turbulent transition and suppress turbulence.

Research in this area was pioneered by Nerushev and Novopashin [2], who performed comparative experiments on laminar–turbulent transition in Hagen–Poiseuille flow in a round tube for nitrogen N_2 and carbon monoxide CO. The thermodynamic and transport properties of these gases are almost identical but the bulk viscosity of CO calculated from data on ultrasound attenuation is several times higher that the similarly calculated value for N_2. It was found in the experiments that, under the same conditions, the transition Reynolds number Re_t in the more viscous gas CO was approximately 10% higher than the corresponding value for N_2.

For some reasons, the validity of the indicated results was questionable. In particular, for the bulk viscosities of the gases used there are different data (see the references in [3]) obtained by measuring relaxation times in shock waves. From these data, which are also given in part in [2], it follows that the difference between the bulk viscosities N_2 and CO is small so that it cannot be responsible for the observed change in Re_t. The fact that in [2] there are no comments on this inconsistency was noted in [4].

Bertolotti [4] employed linear stability theory to numerically study the effect produced by excitation of the internal degrees of freedoms of molecules on laminar–turbulent transition (LTT) in a compressible boundary layer on a plate. Calculations for supersonic airflow have shown that accounting for bulk viscosity lead to an insignificant stabilizing effect, which is manifested in small deformations of the neutral curves for the first and second unstable modes (for the definition of these modes, first introduced in [5], see in [6]). For flow over a plate at Mach numbers ultimately admissible for the Navier–Stokes model and at ratios of bulk and dynamic viscosities realistic for diatomic gases, estimates using linear theory [7] have also shown that bulk viscosity has a weak effect on the value of Re_t.

Nevertheless the results [4, 7] obtained in a linear approximation are not in direct contradiction to experimental data [4]. As is known, linear stability theory satisfactorily describes LTT on a plate, whereas Hagen–Poiseuille flow in a linear approximation is steady-state. At the same time, in [4], the transition to turbulence was observed up to the final nonlinear stage.

[1]Institute of Computational Technologies, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; grigor@ict.nsc.ru. [2]Novosibirsk State Academy of Water Transport, Novosibirsk 630099. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 3, pp. 73–84, May–June, 2008. Original article submitted December 7, 2006; revision submitted March 2, 2007.

To estimate the effect of bulk viscosity on the nonlinear development of perturbations, Grigor'ev and Ershov studied [8] compressible Couette flow perturbed by a Rankine vortex. Despite simplicity, this model adequately describes the evolution of large vortex structures against the background of the carrier shear flow, which is a characteristic element of modern scenarios of transition and generation of developed turbulence [9]. Calculations [8] of such flow using the full Navier–Stokes equations for a viscous heat-conducting gas have shown that, in the realistic range of bulk viscosities, the dissipation effect is rather significant. In this case, the relative change in the rate of damping of the initial vortex perturbation reaches 10%.

Because the calculations of [8] were made on a rather coarse mesh, at least, part of the dissipation effect, which is a few percent, may be attributed to the effect of schematic viscosity. In [10], the model flow [8] was again calculated for a sequence of nested meshes to separate the physical and approximation effects. The calculations using the scheme of [11] with a symmetric approximation of convective derivatives confirmed that the change in the dissipation effect in [8] is almost entirely due to bulk viscosity.

As is known, the bulk viscosity in the Navier–Stokes equations takes into account the relaxation of internal molecular modes during moderate thermal excitation [1]. In a study [12] of the effect of excitation of the lower vibrational levels, the same model flow was calculated within the framework of two-temperature gas dynamics. The energy relaxation of the vibrational mode to equilibrium was described by the Landau–Teller equation. It was shown that, against the background of only the relaxation process with no viscous dissipation, the suppression of the disturbances remained substantial.

At the same time, the results of studies [8, 10, 12] of purely damping perturbations provide only indirect estimates of the extent to which bulk viscosity (relaxation process) influences LTT. Generally, the dependence of the critical Reynolds number of LTT Re_t on bulk viscosity can be obtained on the basis of the energy theory of global hydrodynamic stability [13]. By the global nature of hydrodynamic stability is meant the unboundedness of the amplitudes of the examined perturbations, for which the energy balance equation is derived [8] for the entire flow region. The values of the stability criteria obtained on the basis of this equation, as a rule, has the meaning of limiting lower-bound estimates and are not always close to experimental data. Nevertheless, this approach is currently the only possible method for taking into account the nonlinear stage of loss of stability, though in generalized form, which is necessary in this case.

It should be noted that energy theory remains unsuitable for compressible flows. This is due to the substantial nonlinearity of the full Navier–Stokes equations for a compressible heat-conducting gas (see the comments and references in [13, 14]). All known results of this theory on the stability of incompressible and inhomogeneous liquid flows have been obtained taking into account the solenoidal nature of the admissible velocity fields, which is absent in compressible flows. The difficulties of mathematical nature that arise in the case have not been overcome.

In the present paper, the stability of a compressible Couette flow with a linear velocity profile is studied using energy theory. Some simplifications make it possible to completely solve the corresponding variational problem for this flow and obtain an explicit dependence of $\mathrm{Re}_{\mathrm{cr}}$ on bulk viscosity.

1. Constitutive Equations. The Couette flow stability problem is considered using the Navier–Stokes equations for a compressible viscous heat-conducting gas. The computation domain Ω is a rectangular parallelepiped, whose faces are parallel to the coordinate planes of the Cartesian system (x_1, x_2, x_3) and whose center coincides with the coordinate origin. The impenetrable infinite plates along which the main current is directed are perpendicular to the x_2 axis.

The characteristic nondimensionalizing scales are the channel width L on the x_2 axis, the modulus of the main-flow velocity U_0, the density ρ_0 and temperature T_0 on the impenetrable walls of the channel, the time $\tau_0 = L/U_0$, and the pressure $p_0 = \rho_0 U_0^2$. In the dimensionless variables, the system of equations is written as

$$\frac{d\rho}{dt} + \rho\,\frac{\partial u_i}{\partial x_i} = 0,$$

$$\rho\,\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\mathrm{Re}}\Big(\alpha + \frac{1}{3}\Big)\frac{\partial^2 u_j}{\partial x_i\,\partial x_j},$$

$$\rho\frac{dT}{dt} + \gamma(\gamma-1)\mathrm{M}_0^2 p\,\frac{\partial u_i}{\partial x_i} = \frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}}\,\frac{\partial^2 T}{\partial x_i^2}, \qquad (1)$$

$$\gamma \mathrm{M}_0^2 p = \rho T, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}, \qquad i, j = 1, 2, 3.$$

Here ρ, u_i, T, and p are the density, velocity components, temperature, and gas pressure, respectively; the summation is performed over repeated subscripts. It is assumed that the thermal capacity and dissipation coefficients in system (1) do not depend on temperature and are constant. The parameters included in Eqs. (1) are defined as follows: the coefficient α is equal to the ratio of the bulk viscosity to the shear viscosity ($\alpha = \eta_b/\eta$) and characterizes the degree of nonequilibrium of the internal degrees of freedom of the gas molecules; $\mathrm{M}_0 = U_0/\sqrt{\gamma R T_0}$ is the Mach number of the main flow, $\mathrm{Re} = U_0 L \rho_0/\eta$ is the Reynolds number, $\mathrm{Pr} = \eta c_p/\lambda_0$ is the Prandtl number, R is the gas constant, $\gamma = c_p/c_v$ is the isentropic exponent, c_p are c_v are the specific heats at constant pressure and volume, respectively and λ_0 is the thermal conductivity. In the energy equation, the group of nonlinear terms constituting the so-called dissipation function are omitted. This approximation is widely used in stability problems for compressible flows [5, 6].

Plane Couette flow with a linear velocity profile, which is an exact steady-state solution of system (1), is described by the relations

$$\boldsymbol{U}_s(x_2) = (x_2, 0, 0), \quad T_s(x_2) = \rho_s(x_2) = 1, \quad p_s(x_2) = 1/(\gamma \mathrm{M}_0^2).$$

Representing the instantaneous values of the hydrodynamic quantities of the perturbed flow as

$$\rho = 1 + \rho', \quad u_i = U_{s,i} + u_i', \quad T = 1 + T', \quad p = 1/(\gamma \mathrm{M}_0^2) + p', \tag{2}$$

we write the equations for the perturbations ρ', u_i', T', and p' of the main flow without constraint on their amplitudes:

$$\frac{\partial \rho'}{\partial t} + u_i \frac{\partial \rho'}{\partial x_i} + \rho \frac{\partial u_i'}{\partial x_i} = 0; \tag{3}$$

$$\rho\Big(\frac{\partial u_i'}{\partial t} + u_j' \frac{\partial u_i'}{\partial x_j} + U_{s,j} \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial U_{s,i}}{\partial x_j}\Big) = -\frac{\partial p'}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{1}{\mathrm{Re}}\Big(\alpha + \frac{1}{3}\Big) \frac{\partial^2 u_j'}{\partial x_i \, \partial x_j}; \tag{4}$$

$$\rho\Big(\frac{\partial T'}{\partial t} + u_j' \frac{\partial T'}{\partial x_j} + U_{s,j} \frac{\partial T'}{\partial x_j}\Big) + \gamma(\gamma - 1)\mathrm{M}_0^2 p \frac{\partial u_i'}{\partial x_i} = \frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}} \frac{\partial^2 T'}{\partial x_i^2}; \tag{5}$$

$$\gamma \mathrm{M}_0^2 p' = \rho T' + \rho', \qquad i, j = 1, 2, 3. \tag{6}$$

Equations (3)–(5) do not contain an explicit dependence of the unperturbed flow velocity (2) on the x_2 coordinate lest the form of summation over subscripts be complicated. It is assumed that, for $x_1 = \pm x_0/2$ and $x_3 = \pm z_0/2$, the perturbations of the velocity u_i', density ρ', and pressure p' satisfy the periodic boundary conditions, and on impenetrable boundaries $x_2 = \pm 1/2$, they vanish. For the temperature perturbation T', the following boundary conditions are specified:

$$\frac{\partial T'}{\partial x_1}\Big|_{x_1 = -x_0/2} = \frac{\partial T'}{\partial x_1}\Big|_{x_1 = +x_0/2}, \qquad \frac{\partial T'}{\partial x_2}\Big|_{x_2 = -1/2} = \frac{\partial T'}{\partial x_2}\Big|_{x_2 = +1/2} = 0,$$

$$\frac{\partial T'}{\partial x_3}\Big|_{x_3 = -z_0/2} = \frac{\partial T'}{\partial x_3}\Big|_{x_3 = +z_0/2}.$$

Below, the dimensions of the domain Ω on the periodic (homogeneous) coordinates x_1 and x_3 are equal to the perturbation wavelength on the corresponding coordinate:

$$x_0 = \pi/\beta, \qquad z_0 = \pi/\delta.$$

Here β and δ are the moduli of the projections of the perturbation wave vector $\boldsymbol{k}$ on the x_1 and x_3 axes, respectively.

2. Energy Balance Equations and Functionals. We define the kinetic energy of the perturbations as an integral over the flow region in the form

$$E(t) = \int\limits_{\Omega} \frac{\rho u_i'^2}{2} \, d\Omega.$$

For the evolution of the quantity $E(t)$, from Eqs. (3) and (4), we derive the energy balance equation similarly to [8]. For this, Eqs. (3) and (4) are multiplied by $u_i'^2$ and u_i', respectively, and are combined. On the left side of the resulting relation there is a series of terms in divergent form:

$$\frac{1}{2}\frac{\partial}{\partial t}(\rho u_i'^2)+\frac{1}{2}\frac{\partial}{\partial x_j}(\rho u_i'^2 u_j')+\frac{1}{2}\frac{\partial}{\partial x_j}(\rho u_i'^2)+\rho u_i' u_j'\frac{\partial U_{s,i}}{\partial x_j}$$
$$=-u_i'\frac{\partial p'}{\partial x_i}+\frac{1}{\mathrm{Re}}u_i'\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{1}{\mathrm{Re}}\Big(\alpha+\frac{1}{3}\Big)u_i'\frac{\partial}{\partial x_i}\frac{\partial u_j'}{\partial x_j}. \tag{7}$$

Integration of equality (7) over the domain Ω transforms the divergent terms on the left side to integrals over the boundary, which vanish by virtue of the boundary conditions on the perturbations. The terms on the right side are integrated by parts, and the resulting boundary integrals also vanish. As a result, we have the integral equation

$$\frac{dE}{dt}\equiv\frac{d}{dt}\int\limits_{\Omega}\frac{\rho u_i'^2}{2}\,d\Omega=J_1+J_2-\frac{1}{\mathrm{Re}}(J_3+\alpha J_4). \tag{8}$$

The term

$$J_1=-\int\limits_{\Omega}\rho u_i' u_j'\frac{\partial U_i}{\partial x_j}\,d\Omega$$

describes the energy exchange between the perturbation and the main flow. The integral

$$J_2=\int\limits_{\Omega}p'\frac{\partial u_i'}{\partial x_i}\,d\Omega$$

can be treated as the work in pulsation compression (expansion) of the gas, and the integrals

$$J_3=\int\limits_{\Omega}\Big[\Big(\frac{\partial u_i'}{\partial x_j}\Big)^2+\frac{1}{3}\Big(\frac{\partial u_i'}{\partial x_i}\Big)^2\Big]\,d\Omega,\qquad J_4=\int\limits_{\Omega}\Big(\frac{\partial u_i'}{\partial x_i}\Big)^2\,d\Omega$$

correspond to energy dissipation.

In the above expressions, the signs of the integrals J_1 and J_2 are not determined, whereas J_3 and J_4 are nonnegative. As the Reynolds number Re decreases to a certain value $\mathrm{Re}_{\mathrm{cr}}$, the dissipation terms J_3 and J_4 begin to dominate and the derivative $dE/dt<0$ and any perturbations damp with time. This allows one to formulate a variational problem based on Eq. (8) to estimate the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$, which that corresponds to the condition $dE/dt=0$ and is calculated as the minimum of the functional:

$$\mathrm{Re}_{\mathrm{cr}}=\min\Big(\frac{J_3+\alpha J_4}{J_1+J_2}\Big). \tag{9}$$

From equality (9), it follows that an increase in the bulk viscosity (or the parameter α) leads to an increase in the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$, but to obtain a particular value of $\mathrm{Re}_{\mathrm{cr}}$, it is necessary to solve the variational eigenvalue problem [13].

At the same time, Eq. (8) was derived similarly to the equation for an incompressible fluid [13] and, in this form, it does not explicitly take into account the perturbation features in compressible flows. In particular, unlike for an incompressible fluid, the total perturbation energy in gases, especially in molecular gases should contain not only the kinetic component $E(t)$ but also the internal energy in any form. In addition, Eq. (8) does not contain an explicit dependence on the Mach number M_0. This is due to the fact that Eq. (8) was derived without using the energy equation (5) and the equation of state (6).

The energy balance equation (8) can be transformed as follows. Using equality (2), the continuity equations (3), and the equation of state (6), we write Eq. (5) as

$$p'\frac{\partial u_i'}{\partial x_i}=-\frac{\partial}{\partial t}\Big(\frac{\rho T}{\gamma(\gamma-1)\mathrm{M}_0^2}\Big)-\frac{1}{(\gamma-1)\mathrm{M}_0^2}\frac{\partial}{\partial x_i}\Big(u_i'+\mathrm{M}_0^2 u_i p'-\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial T'}{\partial x_i}\Big). \tag{10}$$

After the substitution of expressions (10) into the integral J_2, the divergent terms vanish, by virtue of the boundary conditions on the perturbations, and, on the left of Eq. (8), we have the time derivative of the integral [15]:

$$E_t(t) = \int\limits_\Omega \rho\Big(\frac{u_i'^2}{2} + \frac{T}{\gamma(\gamma-1)\mathrm{M}_0^2}\Big)\, d\Omega.$$

In view of the chosen nondimensionalization method, it is easy to show that, in the dimensional variables, the term $\rho T/[\gamma(\gamma-1)\mathrm{M}_0^2]$ is the internal energy of the gas in unit volume. Obviously, the energy functional E_t is positive definite. The converted energy balance equation becomes

$$\frac{dE_t}{dt} = \Phi \equiv -\int\limits_\Omega \Big\{(1+\rho')u_i'u_j'\,\frac{\partial U_{s,i}}{\partial x_j} + \frac{1}{\mathrm{Re}}\Big[\Big(\frac{\partial u_i'}{\partial x_j}\Big)^2 + \Big(\alpha + \frac{1}{3}\Big)\Big(\frac{\partial u_i'}{\partial x_i}\Big)^2\Big]\Big\}\, d\Omega. \tag{11}$$

For (11), it is also possible to formulate a variational eigenvalue problem to find the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$. To further simplify Eq. (11), we perform partial separation of the variables and write the dependences of the perturbations of the velocity, density, and temperature on the periodic coordinate x_3 in the form

$$\begin{gathered} u_1' = u_1''(x_1,x_2)\cos(\delta x_3), \quad u_2' = u_2''(x_1,x_2)\cos(\delta x_3), \quad u_3' = u_3''(x_1,x_2)\sin(\delta x_3), \\ \rho' = \rho''(x_1,x_2)\cos(\delta x_3), \qquad T' = T''(x_1,x_2)\cos(\delta x_3). \end{gathered} \tag{12}$$

At $x_1 = \pm\pi/\beta$, the amplitude functions u_i'', ρ'', and T'' satisfy the periodic boundary conditions, and on the impenetrable boundaries $x_2 = \pm 1/2$, they vanish. Using representation (12), in Eq. (11) we perform integration over the variable x_3 in the range $[-\pi/\delta; \pi/\delta]$. As shown in [14], the operations of variation and partial integration over homogeneous coordinates are permutational and a change in their order does not change the original variational problem. As a result, we have

$$\begin{gathered} \frac{dE_t''}{dt} = \Phi'' \equiv -\int\limits_S \Big\{u_1''u_2'' + \frac{1}{\mathrm{Re}}\Big[\Big(\frac{\partial u_1''}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_1''}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_2''}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_2''}{\partial x_2}\Big)^2 \\ + \Big(\frac{\partial u_3''}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_3''}{\partial x_2}\Big)^2 + \delta^2(u_1''^2 + u_2''^2 + u_3''^2) + \Big(\alpha + \frac{1}{3}\Big)\Big(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} + \delta u_3''\Big)^2\Big]\Big\} dS. \end{gathered} \tag{13}$$

From expression (13), it follows that, after transformation (11), the varied functional Φ'' on the right side becomes quadratic in the amplitude functions u_i''.

3. Spectral Problem. Subjecting the functions u_k'' in the functional Φ'' to small smooth variations $u_k''+\delta u_k''$ admitted by the boundary conditions, we distinguish a functional $L(\delta u_k'')$, which is linear in the increment vector and leads to the Euler–Lagrange equations

$$\begin{gathered} \Delta_2 u_1'' + \Big(\alpha + \frac{1}{3}\Big)\frac{\partial}{\partial x_1}\Big(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} + \delta u_3''\Big) = \frac{\mathrm{Re}}{2}\, u_2'', \\ \Delta_2 u_2'' + \Big(\alpha + \frac{1}{3}\Big)\frac{\partial}{\partial x_2}\Big(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} + \delta u_3''\Big) = \frac{\mathrm{Re}}{2}\, u_1'', \\ \Delta_2 u_3'' - \delta\Big(\alpha + \frac{1}{3}\Big)\Big(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} + \delta u_3''\Big) = 0, \end{gathered} \tag{14}$$

where the operator Δ_2 has the form

$$\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \delta^2.$$

System (14) defines the differential eigenvalue problem with the spectral parameter Re.

The velocity pulsation vector $\boldsymbol{u}''$ can be represented as

$$\boldsymbol{u}'' \equiv (u_1'', u_2'', u_3'') = \boldsymbol{v}\exp(i\beta x_1), \tag{15}$$

where $\boldsymbol{v} = (u(x_2), v(x_2), w(x_2))$ is the vector of the perturbation amplitudes, β is the absolute value of the projection of the wave vector onto the x_1 coordinate axis, and i is imaginary unit. Substitution of Eq. (15) into the Euler–Lagrange equations (14) leads to the following system of differential equations for the amplitudes u, v, and w:

$$\frac{d^2u}{dy^2} + i\beta\Big(\alpha + \frac{1}{3}\Big)\frac{dv}{dy} - \Big[\beta^2\Big(\alpha + \frac{4}{3}\Big) + \delta^2\Big]u - \frac{\mathrm{Re}}{2}\, v + i\beta\delta\Big(\alpha + \frac{1}{3}\Big)w = 0,$$

$$\left(\alpha+\frac{4}{3}\right)\frac{d^2v}{dy^2}+i\beta\left(\alpha+\frac{1}{3}\right)\frac{du}{dy}+\delta\left(\alpha+\frac{1}{3}\right)\frac{dw}{dy}-\frac{\mathrm{Re}}{2}\,u-(\beta^2+\delta^2)v=0, \tag{16}$$

$$\frac{d^2w}{dy^2}-\delta\left(\alpha+\frac{1}{3}\right)\frac{dv}{dy}-i\beta\delta\left(\alpha+\frac{1}{3}\right)u-\left[\delta^2\left(\alpha+\frac{4}{3}\right)+\beta^2\right]w=0,$$

$$u\Big|_{y=\pm1/2}=v\Big|_{y=\pm1/2}=w\Big|_{y=\pm1/2}=0.$$

Here and below, the x_2 coordinate is redenoted by y. We note that system (16) is not reduced to a lower-order system by a linear change of variables, as in linear stability theory (cf. [6]); therefore, analytical results can be obtained only in particular cases, which are considered below.

3.1. *Constant Mode* $\beta=\delta=0$. In this case, system (16) becomes

$$\frac{d^2u}{dy^2}-\frac{\mathrm{Re}}{2}\,v=0,\quad \left(\alpha+\frac{4}{3}\right)\frac{d^2v}{dy^2}-\frac{\mathrm{Re}}{2}\,u=0,\quad \frac{d^2w}{dy^2}=0,$$
$$u\Big|_{y=\pm1/2}=v\Big|_{y=\pm1/2}=w\Big|_{y=\pm1/2}=0. \tag{17}$$

The third equation of system (17) is integrated separately and has the general solution

$$w=c_1x_2+c_2,$$

which vanishes identically under zero boundary conditions.

The characteristic equation of the thus abridged system (17) becomes

$$\lambda^4-(\mathrm{Re}/2)^2(\alpha+4/3)^{-1}=0.$$

The roots of this equation are

$$\lambda_{1,2}=\pm a,\qquad \lambda_{3,4}=\pm ia,\qquad a=\sqrt{\mathrm{Re}/2}(\alpha+4/3)^{-1/4}.$$

The general solution of the abridged system (17) is written as

$$\boldsymbol{V}=c_1\boldsymbol{V}_1\,\mathrm{e}^{ax_2}+c_2\boldsymbol{V}_2\,\mathrm{e}^{-ax_2}+c_3\boldsymbol{V}_3\cos(ax_2)+c_4\boldsymbol{V}_4\sin(ax_2),$$

where $\boldsymbol{V}=(u,v)$; $\boldsymbol{V}_k=(u_k,v_k)$ $(k=1,2,3,4)$ are eigenvectors. Using the homogeneous boundary conditions, we obtain $\boldsymbol{V}_1=\boldsymbol{V}_2\equiv0$; the nontrivial solutions are possible in two cases:

$$\boldsymbol{V}_3\neq0,\qquad \boldsymbol{V}_4=0,\qquad \cos(a/2)=0 \tag{18}$$

or

$$\boldsymbol{V}_3=0,\qquad \boldsymbol{V}_4\neq0,\qquad \sin(a/2)=0. \tag{19}$$

As a result, from conditions (18) and (19), it follows that the eigenvalue spectra have the following form, respectively:

$$\mathrm{Re}^{(0)}_{\mathrm{cr},n}=2\pi^2(2n-1)^2(\alpha+4/3)^{1/2},\quad \mathrm{Re}^{(0)}_{s,n}=8\pi^2n^2(\alpha+4/3)^{1/2},\quad n=1,2,3,\ldots\ .$$

The critical value of the Reynolds number $\mathrm{Re}^{(0)}_{\mathrm{cr}}$ is determined as the minimum value of the sets $\mathrm{Re}^{(0)}_{1,n}$ and $\mathrm{Re}^{(0)}_{2,n}$:

$$\mathrm{Re}^{(0)}_{\mathrm{cr}}=\min_{n\in\mathbb{N}}\left(\mathrm{Re}^{(0)}_{\mathrm{cr},n},\mathrm{Re}^{(0)}_{s,n}\right)=2\pi^2(\alpha+4/3)^{1/2}.$$

3.2. *Longitudinal Modes* $\beta\ll1$ *and* $\delta=0$. For $\delta=0$, system (16) reduces to the system

$$\frac{d^2u}{dy^2}+i\beta\left(\alpha+\frac{1}{3}\right)\frac{dv}{dy}-\beta^2\left(\alpha+\frac{4}{3}\right)u-\frac{\mathrm{Re}}{2}\,v=0,$$

$$\left(\alpha+\frac{4}{3}\right)\frac{d^2v}{dy^2}+i\beta\left(\alpha+\frac{1}{3}\right)\frac{du}{dy}-\frac{\mathrm{Re}}{2}\,u-\beta^2v=0, \tag{20}$$

$$\frac{d^2w}{dy^2}-\beta^2w=0,$$

$$u\Big|_{y=\pm 1/2} = v\Big|_{y=\pm 1/2} = w\Big|_{y=\pm 1/2} = 0.$$

The equation for the transverse component in (20) is integrated separately and has the general solution

$$w = c_1\,\mathrm{e}^{\beta x_2} + c_2\,\mathrm{e}^{-\beta x_2}.$$

Substitution of this solution into the zero boundary conditions for w yields the following homogeneous system for arbitrary constants:

$$c_1\,\mathrm{e}^{\beta/2} + c_2\,\mathrm{e}^{-\beta/2} = 0, \qquad c_1\,\mathrm{e}^{-\beta/2} + c_2\,\mathrm{e}^{\beta/2} = 0.$$

From this, it follows that, for $\beta \neq 0$, the solution $w \equiv 0$.

For the thus abridged system (20), the characteristic equations becomes an incomplete quadratic equation, which can be written in standard form [16]

$$\lambda^4 + p\lambda^2 + q\lambda + r = 0, \tag{21}$$

where

$$p = -2\beta^2, \qquad q = i\,\mathrm{Re}\,\beta(1+3\alpha)/(4+3\alpha), \qquad r = \beta^4 - 3\,\mathrm{Re}^2/[4(4+3\alpha)].$$

The roots of Eq. (21) are calculated through the roots of the resolvent cubic equation, which is written in reduced form as

$$z^3 + p_1 z + q_1 = 0,$$

$$p_1 = \frac{3\,\mathrm{Re}^2}{4+3\alpha} - \frac{16}{3}\,\beta^4, \qquad q_1 = \frac{\mathrm{Re}}{4+3\alpha}\,2(9\alpha^2+18\alpha+17)\beta^2 - \frac{128}{27}\,\beta^6.$$

The discriminant of the cubic resolvent is

$$D = (p_1/3)^3 + (q_1/2)^2 > 0.$$

From this it follows that, for arbitrary β, Eq. (21) has two real and two complex conjugate roots. However, because the expressions are cumbersome, their further analysis is generally complicated.

Let us consider the long-wave approximation assuming that $\beta \ll 1$. Generally, the roots of Eq. (21) are calculated from the formulas [16]

$$\lambda_1 = \Big(\sqrt{z_1 + 4\beta^2/3} + \sqrt{z_2 + 4\beta^2/3} + \sqrt{z_3 + 4\beta^2/3}\,\Big)/2,$$

$$\lambda_2 = \Big(\sqrt{z_1 + 4\beta^2/3} - \sqrt{z_2 + 4\beta^2/3} - \sqrt{z_3 + 4\beta^2/3}\,\Big)/2,$$

$$\lambda_3 = \Big(-\sqrt{z_1 + 4\beta^2/3} + \sqrt{z_2 + 4\beta^2/3} - \sqrt{z_3 + 4\beta^2/3}\,\Big)/2,$$

$$\lambda_4 = \Big(-\sqrt{z_1 + 4\beta^2/3} - \sqrt{z_2 + 4\beta^2/3} + \sqrt{z_3 + 4\beta^2/3}\,\Big)/2,$$

where z_k $(k = 1, 2, 3)$ are roots of the resolvent cubic equation. The roots of the characteristic equation (21) are expressed, to within terms of order $O(\beta^4)$, as follows:

$$\lambda_{1,2} = \pm a_1 + ib, \qquad \lambda_{3,4} = \pm i(a_2 \mp b),$$

$$a_{1,2} = \sqrt{\frac{\mathrm{Re}}{2}}\Big(\alpha + \frac{4}{3}\Big)^{-1/4}\Big[1 \pm \frac{\beta^2(7+3\alpha)^2}{36\,\mathrm{Re}}\Big(\alpha + \frac{4}{3}\Big)^{-1/2}\Big], \quad b = \frac{\beta(1+3\alpha)}{6}\Big(\alpha + \frac{4}{3}\Big)^{-1/2}.$$

In the abridged system (20), since the coefficients are complex-valued, the amplitudes of the velocity pulsations u and v are expressed in terms of the real part of the general solution of this system:

$$\mathrm{Re}\,(\boldsymbol{V}) = \mathrm{Re}\,\Big(\sum_{k=1}^{4} c_k \boldsymbol{V}_k\,\mathrm{e}^{\lambda_k}\Big) \tag{22}$$

[$\boldsymbol{V}_k = (u_{k1} + iu_{k2}, v_{k1} + iv_{k2})$ are complex-valued eigenvectors].

Considering each term in (22) separately and taking into account the homogeneous boundary conditions for the component of the eigenvectors for $k = 1, 2$, we obtain the systems

$$q_{k1}\cos(b/2) - q_{k2}\sin(b/2) = 0, \quad q_{k1}\cos(b/2) + q_{k2}\sin(b/2) = 0, \quad q_{kj} = (u_{kj}, v_{kj}).$$

These systems have nontrivial solutions if $\sin b = 0$, which is ruled out in the case $\beta \neq 0$. From this, we have $\boldsymbol{V}_1 = \boldsymbol{V}_2 = 0$. Similarly, for the eigenvector components for $k = 3, 4$, we have homogeneous systems of the form

$$q_{k1}\cos\Big(\frac{a_2 \mp b}{2}\Big) \mp q_{k2}\sin\Big(\frac{a_2 \mp b}{2}\Big) = 0, \qquad q_{k1}\cos\Big(\frac{a_2 \mp b}{2}\Big) \pm q_{k2}\sin\Big(\frac{a_2 \mp b}{2}\Big) = 0,$$

$$q_{kj} = (u_{kj}, v_{kj}),$$

where the upper signs correspond to the case $k = 3$, and the lower signs to $k = 4$. These systems have nontrivial solutions if the following conditions are satisfied:

$$\sin(a_2 \mp b) = 0. \tag{23}$$

Using (23), we obtain the following equations for the eigenvalues Re:

$$x^2 - p_\pm x - s = 0. \tag{24}$$

Here

$$\begin{gathered} x = \sqrt{\mathrm{Re}/2}(\alpha + 4/3)^{-1/4}, \qquad s = \beta^2(7 + 3\alpha)^2/[24(4 + 3\alpha)], \\ p_\pm = \pi n[1 \pm \beta(1 + 3\alpha)(\alpha + 4/3)^{-1/2}/(6\pi n)], \qquad n = 1, 2, 3, \dots\ , \end{gathered} \tag{25}$$

where the upper sign corresponds to the first condition in (23).

The roots of the quadratic equation (24), to within terms of order $O(\beta^3)$, are given by

$$x_1 = \pi n\Big[1 \pm \frac{\beta(1 + 3\alpha)}{6\pi n}\Big(\alpha + \frac{4}{3}\Big)^{-1/2}\Big] + \frac{\beta^2\pi n}{4}\Big[\frac{(1 + 3\alpha)^2}{(6\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1} + \frac{(7 + 3\alpha)^2}{18(\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1}\Big],$$

$$x_2 = -\frac{\beta^2\pi n}{4}\Big[\frac{(1 + 3\alpha)^2}{(6\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1} + \frac{(7 + 3\alpha)^2}{18(\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1}\Big].$$

The root x_2 is not considered further since it corresponds to the out-of-order dependence $\mathrm{Re} \sim O(\beta^4)$, which was neglected above in the expressions for the roots λ_k of the characteristic equation.

The spectra of the eigenvalues Re are given, to within terms of order $O(\beta^3)$, for the root x_1, by the relations

$$\begin{gathered} \mathrm{Re}_n^{(\beta)} = 2\pi^2 n^2\Big(\alpha + \frac{4}{3}\Big)^{1/2}\Big\{\Big[1 \pm \frac{\beta(1 + 3\alpha)}{6\pi n}\Big(\alpha + \frac{4}{3}\Big)^{-1/2}\Big]^2 \\ + \frac{\beta^2}{2}\Big[\frac{(1 + 3\alpha)^2}{(6\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1} + \frac{(7 + 3\alpha)^2}{18(\pi n)^2}\Big(\alpha + \frac{4}{3}\Big)^{-1}\Big]\Big\}, \quad n = 1, 2, 3, \dots\ , \end{gathered} \tag{26}$$

where the plus sign corresponds to the first condition in (23).

From (26), it follows that the minimum Reynolds number $\mathrm{Re}_{\mathrm{cr}}^{(\beta)}$ for long-wave longitudinal modes is

$$\mathrm{Re}_{\mathrm{cr}}^{(\beta)} = 2\pi^2\Big(\alpha + \frac{4}{3}\Big)^{1/2}\Big[1 - \frac{\beta(1 + 3\alpha)}{3\pi}\Big(\alpha + \frac{4}{3}\Big)^{-1/2} + \frac{\beta^2}{72\pi^2}(45\alpha^2 + 102\alpha + 101)\Big(\alpha + \frac{4}{3}\Big)^{-1}\Big].$$

3.3. *Transverse Modes* $\beta = 0$ *and* $\delta \ll 1$. Investigation of these modes is of interest since, for an incompressible Couette flow, the critical Reynolds number the closest to experimental values was obtained for a transverse mode [13]. For $\beta = 0$, system (20) becomes

$$\begin{gathered} \frac{d^2u}{dy^2} - \delta^2 u - \frac{\mathrm{Re}}{2}v = 0, \\ \Big(\alpha + \frac{4}{3}\Big)\frac{d^2v}{dy^2} + \delta\Big(\alpha + \frac{1}{3}\Big)\frac{dw}{dy} - \frac{\mathrm{Re}}{2}u - \delta^2 v = 0, \\ \frac{d^2w}{dy^2} - \delta\Big(\alpha + \frac{1}{3}\Big)\frac{dv}{dy} - \delta^2\Big(\alpha + \frac{4}{3}\Big)w = 0, \end{gathered} \tag{27}$$

$$u\Big|_{y=\pm 1/2} = v\Big|_{y=\pm 1/2} = w\Big|_{y=\pm 1/2} = 0.$$

The characteristic equation of system (27) is written as

$$\lambda^6 - 3\delta^2\lambda^4 + \frac{3}{4}\Big(4\delta^4 - \frac{\mathrm{Re}^2}{4+3\alpha}\Big)\lambda^2 - \delta^2\Big(\delta^4 - \frac{\mathrm{Re}^2}{4}\Big) = 0. \tag{28}$$

The change $z = \lambda^2 - \delta^2$ transforms this equation to the reduced cubic equation

$$z^3 + p_1 z + q_1 = 0,$$

where

$$p_1 = -(\mathrm{Re}\,/2)^2(\alpha + 4/3)^{-1}, \qquad q_1 = \delta^2\,\mathrm{Re}^2(1+3\alpha)(\alpha+4/3)^{-1}/12.$$

The discriminant of this equation

$$D = (p_1/3)^3 + (q_1/2)^2 < 0$$

for the characteristic dependence $\mathrm{Re}\,(\alpha)$ remains negative even in the case $\delta \sim O(1)$. The reduced cubic equation has three real roots defined by the Cardano formulas [16]:

$$z_k = 2\xi^{1/3}\cos\left[(\varphi + 2k\pi)/3\right], \qquad k = 0, 1, 2.$$

Here

$$\xi = \sqrt{-\Big(\frac{p_1}{3}\Big)^3} = \Big[\frac{\mathrm{Re}}{2\sqrt{3}}\Big(\alpha + \frac{4}{3}\Big)^{-1/2}\Big]^3, \quad \cos\varphi = -\frac{q_1}{2\xi} = -\frac{\delta^2\sqrt{3}}{\mathrm{Re}}\,(1+3\alpha)\Big(\alpha + \frac{4}{3}\Big)^{1/2}.$$

Retaining terms of order not higher than $O(\delta^2)$ in the expressions for z_k, we obtain the following expressions for the roots of the characteristic equation (28):

$$\lambda_{1,2} = \pm\sqrt{z_0 + \delta^2} = \pm\delta(\alpha + 4/3)^{1/2},$$

$$\lambda_{3,4} = \pm\sqrt{z_1 + \delta^2} = \pm\sqrt{\mathrm{Re}(\alpha + 4/3)^{-1/2}/2 + \delta^2(5 - 3\alpha)/6},$$

$$\lambda_{5,6} = \pm\sqrt{z_2 + \delta^2} = \pm i\sqrt{\mathrm{Re}(\alpha + 4/3)^{-1/2}/2 - \delta^2(5 - 3\alpha)/6}.$$

Thus, the first four roots λ_k $(k = 1, 2, 3, 4)$ are real, and the roots $\lambda_{5,6}$ are complex conjugate, purely imaginary.

Because the velocity pulsation amplitude vector $\boldsymbol{v}$ is real, it is expressed in terms of the real part of the general solution of system (27):

$$\boldsymbol{v} = \mathrm{Re}\,(\boldsymbol{V}) = \mathrm{Re}\,\Big(\sum_{k=1}^{4} c_k \boldsymbol{V}_k\,\mathrm{e}^{\lambda_k}\Big) \tag{29}$$

[$\boldsymbol{V}_k = (u_{k1} + iu_{k2}, v_{k1} + iv_{k2}, w_{k1} + iw_{k2})$ are complex-valued eigenvectors].

For each term in (29), in view of the homogeneous boundary conditions for the amplitudes, it follows that the eigenvectors corresponding to the real roots are zero:

$$\boldsymbol{V}_1 = \boldsymbol{V}_2 = \boldsymbol{V}_3 = \boldsymbol{V}_4 = 0.$$

For the eigenvector components for $k = 5, 6$, the following homogeneous systems hold:

$$q_{k1}\cos(|\lambda_k|/2) - q_{k2}\sin(|\lambda_k|/2) = 0, \qquad q_{k1}\cos(|\lambda_k|/2) + q_{k2}\sin(|\lambda_k|/2) = 0,$$

$$q_{kj} = (u_{kj}, v_{kj}, w_{kj}).$$

Nontrivial solutions of these systems exist for $\sin|\lambda_k| = 0$. In this case, the eigenvalue spectrum has the form

$$\mathrm{Re}_n^{(\delta)} = 2(\alpha + 4/3)^{1/2}[\pi^2 n^2 + \delta^2(5/3 - \alpha)/2], \qquad n = 1, 2, 3, \ldots\ .$$

Consequently, the critical Reynolds number, as the minimal eigenvalue of the set of eigenvalues for transverse modes, is

$$\mathrm{Re}_{\mathrm{cr}}^{(\delta)} = 2\pi^2(\alpha + 4/3)^{1/2}[1 + \delta^2(5/3 - \alpha)/(2\pi^2)].$$

Conclusions. An energy functional leading to an effectively resolvable variational problem for determining the critical Reynolds number LTT $\mathrm{Re}_{\mathrm{cr}}$ was constructed within the framework of nonlinear energy stability theory for compressible flows.

Asymptotic estimates containing the characteristic dependence $\mathrm{Re}_{\mathrm{cr}} \sim \sqrt{\alpha + 4/3}$ in the main order were obtained for the stability of different modes of Couette compressible gas flow. This implies that, at ratios of the bulk to shear viscosity (parameter α) realistic for diatomic gases, the critical Reynolds number can considerably increase with increasing bulk viscosity. The estimates are consistent with data on the effect of bulk viscosity on the stability of boundary layers on a plate obtained within the framework of linear theory [4, 7], because in near-wall and free shear layers, the LTT mechanism are different.

The asymptotics considered are long-wave approximations. This suggests that the obtained relation describes the effect of bulk viscosity on large-scale vortex structures characteristic of the development of Kelvin–Helmholtz instability.

This work was supported by the Russian Foundation for Basic Research (Grant No. 05-01-00359).

REFERENCES

1. M. A. Leontovoich, "Remarks on the theory of sound attenuation in gases," *Zh. Éksp. Teor. Fiz.*, **6**, No. 6, 561–576 (1936).
2. A. Nerushev and S. Novopashin, "Rotational relaxation and transition to turbulence," *Phys. Lett.*, **A232**, 243–245 (1997).
3. V. M. Zhdanov and M. Ya. Alievskii, *Relaxation Processes and Relaxation in Molecular Gases* [in Russian], Nauka, Moscow (1989).
4. F. B. Bertolotti, "The influence of rotational and vibrational energy relaxation on boundary-layer stability," *J. Fluid Mech.*, **372**, 93–118 (1998).
5. L. M. Mack, "Boundary-layer stability theory," Report No. 900-277, Rev. A, Jet Propulsion Laboratory, Pasadena (1969).
6. S. A. Gaponov and A. A. Maslov, *Development of Perturbations in Compressible Flows* [in Russian], Nauka, Novosibirsk (1980).
7. Yu. N. Grigor'ev and I. V. Ershov, "On the effect of rotational relaxation on laminar–turbulent transition," Abstracts Conf. Dedicated to the 40-year Anniversary of the Moscow State University (Moscow, November 22–26, 1999), Izd. Mosk. Gos. Univ., Moscow (1999), pp. 65–66.
8. Yu. N. Grigor'ev and I. V. Ershov "Relaxation-induced suppression of vortex disturbances in a molecular gas," *J. Appl. Mech. Tech. Phys.*, **44**, No. 4, 471–481 (2003).
9. F. K. Browand and Chih Ming Ho, "The mixing layer: an example of quasi two-dimensional turbulence, *J. Mecanique Teor. Appl.*, Spec. number, 99–120 (1983).
10. Yu. N. Grigor'ev, I. V. Ershov, K. V. Zyryanov, and A. V. Sinyaya, "Numerical simulation of the effect of bulk viscosity on a sequence of nested meshes," *Vychisl. Tekhnol.*, **11**, No. 3, 36–49 (2006).
11. V. M. Kovenya and N. N. Yanenko, *Splitting Method in Problems of Gas Dynamics* [in Russian], Nauka, Novosibirsk (1981).
12. Yu. N. Grigor'ev, I. V. Ershov, and E. E. Ershova, "Influence of vibrational relaxation on the pulsation activity in flows of an exited diatomic gas," *J. Appl. Mech. Tech. Phys.*, **45**, No. 3, 321–327 (2004).
13. D. Joseph, *Stability of Fluid Motions*, Springer-Verlag, Berlin–Heidelberg (1976).
14. M. A. Gol'dshtik and V. N. Shtern, *Hydrodynamic Stability and Turbulence* [in Russian], Nauka, Novosibirsk (1977).
15. Yu. N. Grigor'ev, "On the energetic stability theory of compressible flows," *Vychisl. Tekhnol.*, **11**, Special issue, 55–62 (2006).
16. I. N. Bronshtein and K. A. Semendyaev, *Reference Book on Mathematics* [in Russian], Nauka, Moscow (1986).